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Note

A characteristic polynomial for rooted mixed graphs

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Abstract

An arborescence in a rooted mixed graph Ω_r is a tree in which every directed edge is directed away from the root. The collection of arborescences in a rooted mixed graph forms the feasible sets of a greedoid called the branching greedoid of Ω_r. We give a graph theoretical characterization of the characteristic polynomial of a mixed branching greedoid.

Keywords: Greedoid; Rooted mixed graphs; Characteristic polynomial

1. Introduction

The characteristic polynomial of a greedoid was introduced in [1] by Gary Gordon and Elizabeth McMahon as a generalization of the matroid characteristic polynomial. Applying this polynomial to a greedoid, called a directed branching greedoid, defined on a rooted directed graph, in [2], they gave a characterization of this polynomial in terms of the rooted directed graph. However, they did not characterize this polynomial for undirected graphs. They gave partial results related to a class of undirected graphs called fan graphs.

We generalize the definition of directed branching greedoids to rooted mixed graphs, then give a graph-theoretical characterization for the characteristic polynomial of such greedoids. We apply this result to three classes of undirected graphs to determine the characteristic polynomial for the branching greedoids of complete graphs, fan graphs, and wheel graphs.

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2. Definitions

We first define the required greedoid and graph theory terminology. We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$ for any non-negative integer n.

A mixed graph Ω consists of a set $V(\Omega)$ of vertices, a set $U(\Omega)$ of undirected edges, and a set $D(\Omega)$ of directed edges. Each undirected edge is associated to a pair of vertices and each directed edge $e \in D(\Omega)$ is associated to an ordered pair of vertices (v_1, v_2). A directed edge e is said to be directed from v_1 to v_2. If $D(\Omega) = \emptyset$, then Ω is a graph, and if $U(\Omega) = \emptyset$, then Ω is a directed graph. We use $E(\Omega) = U(\Omega) \cup D(\Omega)$ to denote the set of edges of Ω. If there is a distinguished vertex, called the root, then Ω is a rooted mixed graph. If Ω is a mixed graph with root r, we denote this by Ω_r. If no ambiguity will occur, we will use V, E, D, and U to denote the appropriate sets.

A walk in Ω is a sequence $(v_0, e_1, \ldots, v_{n-1}, e_n, v_n)$ of alternating vertices and edges with $n > 0$, and with edge e_i incident to the vertices v_{i-1} and v_i for all $i \in [n-1]$. A path P in Ω is a walk in which $v_i \neq v_j$ for all $i \neq j$. A circle is a walk such that $v_0 = v_n$ and $v_i \neq v_j$ for all other pairs of vertices. A connected subgraph of Ω is a tree if it contains no circles. A directed path P in Ω is a path such that every edge e_i in $D(P)$ is directed from v_{i-1} to v_i.

An arborescence F in Ω_r is a tree in Ω_r containing r with the property that, for every vertex $v \in V(F)$ with $v \neq r$, the path in F from r to v is directed. For a rooted mixed graph Ω_r, we define $\mathscr{F}(\Omega_r)$ by $\mathscr{F}(\Omega_r) = \{E(F) : F \text{ is an arborescence in } \Omega_r\}$. In a typical abuse of notation, we will use F to denote both an arborescence and its edge set.

A set E and a non-empty collection of subsets $\mathscr{F}$ form a greedoid $G = (E, \mathscr{F})$ if:

(1) For each non-empty $F \in \mathscr{F}$, there exists an $e \in F$ such that $F \backslash e \in \mathscr{F}$.
(2) If $F_1, F_2 \in \mathscr{F}$ and $|F_1| > |F_2|$ then there exists an $e \in F_1 \backslash F_2$ with $F_2 \cup e \in \mathscr{F}$.

The sets $F \in \mathscr{F}(G)$ are called the feasible sets of G. If $e \in E$ is not in any feasible subset, then e is called a loop of G. A greedoid is normal if it contains no loops. Given $S \subseteq E$, the rank of S, denoted $r_G(S)$, is defined by:

$$r_G(S) = \max\{|F| \; : \; F \in \mathscr{F}; \; F \subseteq S\}.$$

If there is no confusion about the greedoid under discussion, we will use $r(S)$ to denote the rank function.

We now define the operations of deletion and contraction for greedoids. Given a set $S \subseteq E$, we define the deletion of S as $G \backslash S = (E \backslash S, \mathscr{F} \backslash S)$ where $\mathscr{F} \backslash S = \{F \subseteq E \backslash S : F \in \mathscr{F}\}$. Given a feasible set F, we define the contraction of F as $G/F = (E \backslash F, \mathscr{F}/F)$, where $\mathscr{F}/F = \{H \subseteq E \backslash F : H \cup F \in \mathscr{F}\}$. If we are deleting or contracting a single element e, we write $G \backslash e$ and G/e to denote $G \backslash \{e\}$ and $G/\{e\}$.

Given two greedoids $G_1 = (E_1, \mathscr{F}_1)$ and $G_2 = (E_2, \mathscr{F}_2)$ with $E_1 \cap E_2 = \emptyset$, we define the direct sum of G_1 and G_2 to be $G_1 \oplus G_2 = (E_1 \cup E_2, \mathscr{F}_1 \sqcup \mathscr{F}_2)$, where $\mathscr{F}_1 \sqcup \mathscr{F}_2 = \{F_1 \cup F_2 \; : \; F_1 \in \mathscr{F}_1 \text{ and } F_2 \in \mathscr{F}_2\}$.

3. Mixed branching greedoids

We first show that, given a rooted mixed graph Ω_r, the pair

$$G(\Omega_r) = (E(\Omega), \mathscr{F}(\Omega_r))$$

forms a greedoid.

Theorem 1. *If Ω_r is a rooted mixed graph with root r, then $G(\Omega_r)$ is a greedoid.*

Proof. Since $\emptyset \in \mathscr{F}(\Omega_r)$, $\mathscr{F}(\Omega_r)$ is not empty.

Suppose $F \in \mathscr{F}(\Omega_r)$ with $F \neq \emptyset$. There exists a vertex $v \neq r$ which is adjacent to only one edge $e \in F$. Thus $F \backslash e$ is also connected, and is therefore an arborescence. This implies that $F \backslash e \in \mathscr{F}(\Omega_r)$.

Suppose $F_1, F_2 \in \mathscr{F}(\Omega_r)$ with $|F_1| > |F_2|$. This implies that there exists a vertex v which is incident to an edge in F_1 and not incident to any edge in F_2. Let $P \subseteq F_1$ be the edge set of a path from r to v. Suppose that $P = \{e_1, \ldots, e_n\}$ and that $P_i = \{e_1, \ldots, e_i\}$ is also the edge set of a path for all $i \in [n]$. Let $j = \max\{i : P_i \subseteq F_2\}$. Now, $j < n$ since v_n is not incident to any edge in F_2. Thus e_{j+1} is incident to a vertex which is not incident to any edge in F_2, and is incident to an edge in $F_1 \cap F_2$. Thus $F_2 \cup e_{j+1} \in \mathscr{F}(\Omega_r)$. $\square$

The greedoid introduced in the previous theorem is the branching greedoid of Ω_r and is called a mixed branching greedoid. If Ω is an undirected graph, then $G(\Omega_r)$ is called an undirected branching greedoid [3], while if Ω is a directed graph, then $G(\Omega_r)$ is called a directed branching greedoid [3].

4. The characteristic polynomial

In [1] Gordon and McMahon defined the characteristic polynomial of a greedoid G by

$$p(G; \lambda) = (-1)^{r(G)} \sum_{S \subseteq E} (-\lambda)^{r(G)-r(S)}(-1)^{|S|-r(S)}.$$

They also showed that the characteristic polynomial has the following properties.

Proposition 2 (*Gordon and McMahon [1, Propositions 3 and 4]*). *Let G, G_1, and G_2 be greedoids, and let $\{e\}$ be a feasible set in G. Then*

(1) $p(G; \lambda) = \lambda^{r(G)-r(G\backslash e)} p(G\backslash e; \lambda) - p(G/e; \lambda)$.
(2) $p(G_1 \oplus G_2; \lambda) = p(G_1; \lambda) p(G_2; \lambda)$.

Furthermore, they showed that

Lemma 3. *If G is a greedoid with a loop e, then $p(G; \lambda) = 0$.*

Given a mixed graph Ω, an orientation o of Ω is an application of a direction to each undirected edge in Ω. Given an orientation o, $o(\Omega)$ will denote the directed graph obtained by applying o to Ω. Let $G_o = (E(\Omega), \mathscr{F}(o(\Omega)_r))$. Lemma 3 implies that if e is a loop in G_o, then $p(G_o, \lambda) = 0$. Let $\mathscr{O}(\Omega)$ be the set of all orientations of Ω.

The orientations of the mixed graph determine the characteristic polynomial, as the following proposition shows.

Proposition 4. *Let G be the mixed branching greedoid of a rooted mixed graph Ω_r. Then*

$$p(G; \lambda) = \sum_{o \in \mathscr{O}(\Omega)} p(G_o; \lambda).$$

Prior to proving this proposition, we need to define a leaf. A leaf e is a feasible edge such that $G = (G \backslash e) \oplus (\{e\}, \{\emptyset, \{e\}\})$. The proof of Proposition 4.4 in [2], applied to mixed graphs, implies:

Lemma 5. *For $e \in E$ with $\{e\} \in \mathscr{F}$, if e is not a leaf, then $p(G; \lambda) = p(G \backslash e; \lambda) - p(G/e; \lambda)$.*

Proof. We induct on $|E|$. Suppose $|E| = 0$. Since there are no undirected edges, there is only one orientation $o \in \mathscr{O}(\Omega)$, and, in fact, $G_o = G$. Thus

$$p(G; \lambda) = \sum_{o \in \mathscr{O}(\Omega)} p(G_o, \lambda).$$

Suppose $|E| > 0$.

We consider the two possibilities. First assume e is a leaf. This implies:

$$\begin{aligned} p(G; \lambda) &= p(e; \lambda) p(G \backslash e; \lambda) \\ &= \sum_{o \in \mathscr{O}(\Omega \backslash e)} p(e; \lambda) p((G \backslash e)_o; \lambda) \\ &= \sum_{o \in \mathscr{O}(\Omega)} p(G_o; \lambda), \end{aligned}$$

where the second equality comes from the induction hypothesis and the last equality arises from the fact that, unless e is directed away from the root, the characteristic polynomial is 0.

Assume that e is not a leaf. Notice that $\mathscr{O}(\Omega \backslash e) = \mathscr{O}(\Omega / e)$ since the orientation is being applied to all edges besides e in either case.

$$\begin{aligned} p(G; \lambda) &= p(G \backslash e; \lambda) - p(G/e; \lambda) \\ &= \sum_{o \in \mathscr{O}(\Omega \backslash e)} p((G \backslash e)_o; \lambda) - \sum_{o \in \mathscr{O}(\Omega / e)} p((G/e)_o; \lambda) \\ &= \sum_{o \in \mathscr{O}(\Omega \backslash e)} p((G \backslash e)_o; \lambda) - p((G/e)_o; \lambda) \\ &= \sum_{o \in \mathscr{O}(\Omega)} p(G_o; \lambda). \end{aligned}$$

where, as above, the second equality comes from the induction hypothesis and the last equality arises from the fact that there is a unique orientation of e, for which e is not a loop. $\square$

In a directed graph, a sink is a vertex that has every incident edge directed towards it, while a source is a vertex that has every incident edge directed away from it. In [2], Gordon and McMahon described the characteristic polynomial of a directed branching greedoid in terms of sinks.

Proposition 6 (*Gordon and McMahon [2, Theorem]*). *Let G be a normal directed branching greedoid of a rooted digraph Λ_r. If s is the number of sinks in Λ, then*

$$p(G;\lambda)=\begin{cases}0 & \textit{if } \Lambda \textit{ contains a directed cycle},\\ (-1)^{r(G)}(1-\lambda)^{s} & \textit{otherwise}.\end{cases}$$

Combining this with Proposition 4, we obtain the main theorem.

Theorem 7. *Let Ω_r be a rooted mixed graph. Let $\mathscr{A}$ be the collection of orientations o of Ω without directed cycles, with r as the unique source, and with G_o normal. For $o \in \mathscr{A}$, let $s(o)$ be the number of sinks in $o(\Omega)$. Then*

$$p(G(\Omega_r);\lambda)=\begin{cases}0 & \textit{if } \mathscr{A}=\emptyset,\\ (-1)^{r(G(\Omega_r))}\sum_{o\in\mathscr{A}}(1-\lambda)^{s(o)} & \textit{otherwise}.\end{cases}$$

As a corollary, if we define $\mathscr{A}_{\mathrm{O}}=\{o \in \mathscr{A} \mid s(o) \text{ is odd}\}$ and $\mathscr{A}_{\mathrm{E}}=\{o \in \mathscr{A} \mid s(o) \text{ is even}\}$, then

Corollary 8. *For any rooted mixed graph Ω_r,*

$$p(G(\Omega_r);2)=|\mathscr{A}_{\mathrm{E}}|-|\mathscr{A}_{\mathrm{O}}|.$$

5. Examples

We end by determining the characteristic polynomial for three classes of graphs.

5.1. Complete graphs

The complete graph K_n has n vertices and an edge between every pair of vertices. Letting r be any vertex, we have

$$p(G(K_n);\lambda)=(-1)^{n-1}\sum_{o\in\mathscr{A}}(1-\lambda)^{s(o)}.$$

Reordering these terms by exponent, we obtain

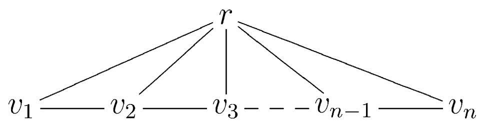
$$p(G(K_n);\lambda)=(-1)^{n-1}\sum_{k=1}^{n}a(n,k)(1-\lambda)^{k},$$

where $a(n, k)$ is the number of orientations of K_n, without directed cycles, that have k sinks. Since such an orientation corresponds to a permutation of the n vertices of which the source is the first member and the sink is the last, the permutation begins with r and the orientation has exactly one sink. Thus, there are $(n-1)!$ possible orientations. This implies that

$$p(G(K_n); \lambda) = (-1)^{n-1}(n-1)!(1-\lambda).$$

5.2. Fan graphs

The fan graph F_n is a path of n vertices together with an additional vertex that is adjacent to every vertex in the path. Letting the additional vertex be the root, F_n is:



We know that

$$p(G(F_n); \lambda) = (-1)^n \sum_{k=1}^{n} b(n, k)(1-\lambda)^k,$$

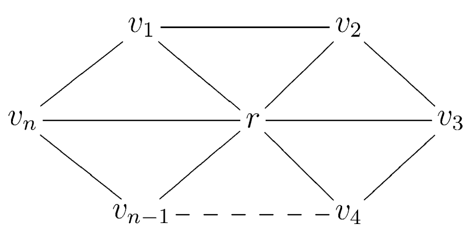
where $b(n, k)$ is the number of orientations of a path with n vertices containing exactly k sinks. The sinks and sources must alternate in the path, starting and ending with a sink. There are $2k-1$ vertices to be designated out of a total of n vertices, so $b(n, k) = \binom{n}{2k-1}$. Thus

$$p(G(F_n); \lambda) = (-1)^n \sum_{k=1}^{n} \binom{n}{2k-1} (1-\lambda)^k.$$

See [2] for other results relating to the characteristic polynomial of fan graphs.

5.3. Wheel graphs

Another class of rooted undirected graphs is the class of wheel graphs, W_n. W_n consists of a circle of n vertices with an additional vertex incident to all of the other vertices. Letting the additional vertex be the root, W_n is:



The characteristic polynomial of W_n is

$$p(G(W_n); \lambda) = (-1)^n \sum_{k=1}^{n} c(n, k)(1-\lambda)^k,$$

where $c(n, k)$ is the number of orientations of a circle graph with n vertices that contain exactly k sinks. Once again, the sources and sinks must alternate, starting at v_1, and there must be exactly k sources and k sinks. Since we can start with either a source or a sink, $c(n, k) = 2\binom{n}{2k}$. Thus

$$p(G(W_n); \lambda) = (-1)^n 2 \sum_{k=1}^{n} \binom{n}{2k} (1-\lambda)^k.$$

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